Nature of the α effect in magnetohydrodynamics

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It is shown that the α effect of mean-field magnetohydrodynamics, which consists of the generation of a mean electromotive force along the mean magnetic field by turbulently fluctuating parts of velocity and magnetic field, is equivalent to the simultaneous generation of both turbulent and mean-field magnetic helicities, the generation rates being equal in magnitude and opposite in sign. In the particular case of statistically stationary and homogeneous fluctuations this implies that the α effect can increase the energy in the mean magnetic field only under the condition that magnetic helicity is also accumulated there.

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I. INTRODUCTION

In order to explain the origin of the cosmical magnetic fields, the theory of the turbulent dynamo has been developed [1–4]. The central mechanism in this theory is the α effect, namely the generation of a mean electromotive force (emf) along a mean, or large-scale, magnetic field by turbulently fluctuating, or small-scale, parts of velocity and magnetic field. The effect has also been invoked to explain the plasma behavior in fusion experiments, in particular, in the reversed field pinch (RFP) [5–8].

Let the evolution of the magnetic field be described by the induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}, \qquad (1)$$

with $\boldsymbol{B}$ and $\boldsymbol{v}$ denoting magnetic induction and fluid velocity and η a constant and positive magnetic diffusivity, $\eta = (\mu_0 \sigma)^{-1}$, σ being the electrical conductivity and μ_0 the magnetic permeability in a vacuum. By splitting up velocity and magnetic fields into mean and fluctuating parts according to

$$\boldsymbol{v} = \langle \boldsymbol{v} \rangle + \boldsymbol{v}', \qquad \boldsymbol{B} = \langle \boldsymbol{B} \rangle + \boldsymbol{B}', \qquad (2)$$

with angular brackets denoting ensemble averages and primes the corresponding residuals, and subsequent averaging of Eq. (1), an equation for the temporal evolution of $\langle \boldsymbol{B} \rangle$ is obtained, namely

$$\frac{\partial \langle \boldsymbol{B} \rangle}{\partial t} = \boldsymbol{\nabla} \times (\langle \boldsymbol{v} \rangle \times \langle \boldsymbol{B} \rangle) + \eta \nabla^2 \langle \boldsymbol{B} \rangle + \boldsymbol{\nabla} \times \mathcal{E}, \qquad (3)$$

where

$$\mathcal{E} = \langle \boldsymbol{v}' \times \boldsymbol{B}' \rangle. \qquad (4)$$

$\mathcal{E}$ is the mean emf caused by the fluctuations. It is a major result of traditional kinematic turbulent-dynamo theory that the presence of kinetic and magnetic helicities is favorable for an α effect, i.e., a nonvanishing component $\mathcal{E}_{\parallel} = \alpha \langle \boldsymbol{B} \rangle$ of $\mathcal{E}$ along $\langle \boldsymbol{B} \rangle$. The densities per unit volume of kinetic, magnetic, and current helicity are defined by

$$H_K = \boldsymbol{v} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}),\ H_M = \boldsymbol{A} \cdot \boldsymbol{B},\ H_C = \boldsymbol{B} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B}), \qquad (5)$$

where $\boldsymbol{A}$ is a vector potential for $\boldsymbol{B}$. In particular, evidence has been found, from statistical-mechanics arguments as well as from numerical simulations [9–14], that magnetic helicity, if present at small scales, cascades to large scales and that this leads to an inverse energy cascade, i.e., to a turbulent-dynamo effect. In the present paper the α effect is interpreted as a mechanism that generates simultaneously small-scale and large-scale magnetic helicities of opposite sign, which seems compatible with the helicity-cascade concept.

II. α EFFECT AND THE GENERATION OF MAGNETIC HELICITY

From Maxwell's equation $\partial \boldsymbol{B}/\partial t = -\boldsymbol{\nabla} \times \boldsymbol{E}$ ($\boldsymbol{E}$ denoting the electric field) and $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ we have

$$\boldsymbol{\nabla} \times \left(\frac{\partial \boldsymbol{A}}{\partial t} + \boldsymbol{E} \right) = \boldsymbol{0} \qquad (6)$$

and consequently

$$\frac{\partial \boldsymbol{A}}{\partial t} = -\boldsymbol{E} + \boldsymbol{\nabla} \phi_{\boldsymbol{A}}, \qquad (7)$$

with some scalar function $\phi_{\boldsymbol{A}}$ depending on the gauge of $\boldsymbol{A}$. For the rate of change of the magnetic helicity we then find

$$\begin{aligned} \frac{\partial}{\partial t}(\boldsymbol{A} \cdot \boldsymbol{B}) &= -\boldsymbol{E} \cdot \boldsymbol{B} + \boldsymbol{\nabla} \phi_{\boldsymbol{A}} \cdot \boldsymbol{B} - \boldsymbol{A} \cdot (\boldsymbol{\nabla} \times \boldsymbol{E}) \\ &= -2\boldsymbol{E} \cdot \boldsymbol{B} - \boldsymbol{\nabla} \cdot (\boldsymbol{E} \times \boldsymbol{A}) + \boldsymbol{\nabla} \phi_{\boldsymbol{A}} \cdot \boldsymbol{B} \\ &= -2\boldsymbol{E} \cdot \boldsymbol{B} - \boldsymbol{\nabla} \cdot (\boldsymbol{E} \times \boldsymbol{A} + \phi_{\boldsymbol{A}} \boldsymbol{B}). \end{aligned} \qquad (8)$$

The first term on the right-hand side of the last of Eqs. (8) is due to the local dissipation or generation, respec-

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of magnetic helicity, while the divergence term describes the flow of magnetic helicity [15].

The mean value of H_M can be written as the sum of two contributions resulting from the mean and fluctuating magnetic fields, respectively, namely

$$\langle H_M \rangle = H_M^{\rm mean} + H_M^{\rm fluc}, \tag{9}$$

with

$$H_M^{\rm mean} = \langle \boldsymbol{A} \rangle \cdot \langle \boldsymbol{B} \rangle, \quad H_M^{\rm fluc} = \langle \boldsymbol{A}' \cdot \boldsymbol{B}' \rangle. \tag{10}$$

By virtue of Eq. (8) we can write

$$\frac{\partial H_M^{\rm mean}}{\partial t} = -2\langle \boldsymbol{E} \rangle \cdot \langle \boldsymbol{B} \rangle + \left(\frac{\partial H_M^{\rm mean}}{\partial t} \right)_{\rm transport} \tag{11}$$

and

$$\frac{\partial H_M^{\rm fluc}}{\partial t} = -2\langle \boldsymbol{E}' \cdot \boldsymbol{B}' \rangle + \left(\frac{\partial H_M^{\rm fluc}}{\partial t} \right)_{\rm transport} \tag{12}$$

for the temporal evolutions of the two parts of $\langle H_M \rangle$. The transport terms vanish, for instance, in a homogeneous turbulence (where all means are independent of position).

From Ohm's law

$$\boldsymbol{E} = \frac{\boldsymbol{j}}{\sigma} - \boldsymbol{v} \times \boldsymbol{B}, \tag{13}$$

where $\boldsymbol{j}$ is the electrical current density and $\mu_0 \boldsymbol{j} = \boldsymbol{\nabla} \times \boldsymbol{B}$, we find

$$\langle \boldsymbol{E} \rangle = \eta \boldsymbol{\nabla} \times \langle \boldsymbol{B} \rangle - \langle \boldsymbol{v} \rangle \times \langle \boldsymbol{B} \rangle - \mathcal{E}. \tag{14}$$

and

$$\boldsymbol{E}' = \eta \boldsymbol{\nabla} \times \boldsymbol{B}' - \langle \boldsymbol{v} \rangle \times \boldsymbol{B}' - \boldsymbol{v}' \times \langle \boldsymbol{B} \rangle - \boldsymbol{v}' \times \boldsymbol{B}' + \mathcal{E}. \tag{15}$$

Equations (11) and (12) then become

$$\frac{\partial H_M^{\rm mean}}{\partial t} = -2\eta \boldsymbol{\nabla} \times \langle \boldsymbol{B} \rangle \cdot \langle \boldsymbol{B} \rangle + 2\mathcal{E} \cdot \langle \boldsymbol{B} \rangle + \left(\frac{\partial H_M^{\rm mean}}{\partial t} \right)_{\rm transport} \tag{16}$$

and

$$\frac{\partial H_M^{\rm fluc}}{\partial t} = -2\eta \langle \boldsymbol{\nabla} \times \boldsymbol{B}' \cdot \boldsymbol{B}' \rangle - 2\mathcal{E} \cdot \langle \boldsymbol{B} \rangle + \left(\frac{\partial H_M^{\rm fluc}}{\partial t} \right)_{\rm transport} . \tag{17}$$

The last two equations show that the α effect corresponds to the simultaneous generation of magnetic helicities in the mean field and in the fluctuations, the generation rates being equal in magnitude and opposite in sign. The mean total magnetic helicity, which is an invariant of ideal magnetohydrodynamics, is not influenced by the α effect. This may equally be considered as a transfer of magnetic helicity between the fluctuating (or small-scale) and the mean (or large-scale) fields, mediated by the α effect.

Consider now a situation in which the magnetic fluctuations are statistically stationary. Actually it is assumed throughout traditional turbulent-dynamo theory (cf. e.g., Ref. [1], Sec. 2.4.) that the magnetic fluctuations have settled down to a statistically stationary state. If then, furthermore, the fluctuations are homogeneous, according to Eq. (17) the α-effect parameter α is connected to the mean current helicity of the fluctuations by (see also Refs. [16–19])

$$\alpha \stackrel{\rm def}{=} \frac{\mathcal{E} \cdot \langle \boldsymbol{B} \rangle}{\langle \boldsymbol{B} \rangle^2} = -\frac{\eta}{\langle \boldsymbol{B} \rangle^2} \langle \boldsymbol{B}' \cdot (\boldsymbol{\nabla} \times \boldsymbol{B}') \rangle. \tag{18}$$

Let us next examine under which conditions there is a turbulent-dynamo effect, i.e., under which conditions the turbulent emf increases the energy in the mean magnetic field. For that purpose we assume $\langle \boldsymbol{v} \rangle = \boldsymbol{0}$, since we are not interested in the dynamo action of the mean flow. For the change of the mean-field magnetic energy density we find

$$\begin{aligned} \frac{\partial}{\partial t} \frac{\langle \boldsymbol{B} \rangle^2}{2} &= -\langle \boldsymbol{B} \rangle \cdot (\boldsymbol{\nabla} \times \langle \boldsymbol{E} \rangle) \\ &= -\langle \boldsymbol{E} \rangle \cdot (\boldsymbol{\nabla} \times \langle \boldsymbol{B} \rangle) + \boldsymbol{\nabla} \cdot (\text{Poynting flux}) \\ &= -\eta (\boldsymbol{\nabla} \times \langle \boldsymbol{B} \rangle)^2 + \mathcal{E} \cdot (\boldsymbol{\nabla} \times \langle \boldsymbol{B} \rangle) \\ &\quad + \boldsymbol{\nabla} \cdot (\text{Poynting flux}), \end{aligned} \tag{19}$$

where in the last step Eq. (14) (with $\langle \boldsymbol{v} \rangle = \boldsymbol{0}$) has been used. Thus, it is a component of $\mathcal{E}$ along (the positive direction of) $\boldsymbol{\nabla} \times \langle \boldsymbol{B} \rangle$, rather than one along $\langle \boldsymbol{B} \rangle$, which is necessary for a turbulent dynamo. If we write $\mathcal{E} = \alpha \langle \boldsymbol{B} \rangle + \mathcal{E}_\perp$, where the first term corresponds to the α effect and $\mathcal{E}_\perp$ is the component of $\mathcal{E}$ perpendicular to $\langle \boldsymbol{B} \rangle$, then $\mathcal{E} \cdot (\boldsymbol{\nabla} \times \langle \boldsymbol{B} \rangle) = \alpha \langle \boldsymbol{B} \rangle \cdot (\boldsymbol{\nabla} \times \langle \boldsymbol{B} \rangle) + \mathcal{E}_\perp \cdot (\boldsymbol{\nabla} \times \langle \boldsymbol{B} \rangle)$. The α effect contributes to the growth of the mean magnetic field if $\alpha(\boldsymbol{\nabla} \times \langle \boldsymbol{B} \rangle \cdot \langle \boldsymbol{B} \rangle) > 0$ or, equivalently [see the definition of α in Eq. (18)], $\mathcal{E} \cdot \langle \boldsymbol{B} \rangle (\boldsymbol{\nabla} \times \langle \boldsymbol{B} \rangle \cdot \langle \boldsymbol{B} \rangle) > 0$. For $\mathcal{E} \cdot \langle \boldsymbol{B} \rangle (\boldsymbol{\nabla} \times \langle \boldsymbol{B} \rangle \cdot \langle \boldsymbol{B} \rangle) < 0$ the α effect lowers the mean-field energy.

As is seen from Eq. (18) and the above discussion, by α effect we simply mean a nonvanishing component of $\mathcal{E}$ along $\langle \boldsymbol{B} \rangle$. Very often $\mathcal{E}$ is expanded as $\mathcal{E}_i = a_{ij} \langle B_j \rangle + b_{ijk} \frac{\partial \langle B_j \rangle}{\partial x_k} + \ldots$, which takes the form $\mathcal{E} = \tilde{\alpha} \langle \boldsymbol{B} \rangle - \beta \boldsymbol{\nabla} \times \langle \boldsymbol{B} \rangle + \ldots$ in the isotropic case. Then the notion α effect refers only to the first term of the expansion, while the second term, the so-called β effect, is interpreted as a turbulent diffusivity. Indeed a pure β effect with $\beta > 0$ corresponds to a turbulent emf with $\mathcal{E} \cdot (\boldsymbol{\nabla} \times \langle \boldsymbol{B} \rangle) < 0$, which, according to our above consideration, lowers the energy of the mean magnetic field.

In the special case of $\mathcal{E} \cdot \langle \boldsymbol{B} \rangle = 0$ (no α effect) there is a pure "pumping effect" (see Ref. [1], Sec. 7.2.). Then $\mathcal{E}$ admits a representation $\mathcal{E} = \boldsymbol{g} \times \langle \boldsymbol{B} \rangle$ with some vector field $\boldsymbol{g}$, which implies $\partial \langle \boldsymbol{B} \rangle / \partial t = \boldsymbol{\nabla} \times [(\langle \boldsymbol{v} \rangle + \boldsymbol{g}) \times \langle \boldsymbol{B} \rangle] + \eta \nabla^2 \langle \boldsymbol{B} \rangle$, showing that the turbulent emf acts like an additional mean flow $\boldsymbol{g}$.

Consider again the case of statistically stationary and homogeneous fluctuations. The condition for a dynamo action of the α effect, $\mathcal{E} \cdot \langle \boldsymbol{B} \rangle (\boldsymbol{\nabla} \times \langle \boldsymbol{B} \rangle \cdot \langle \boldsymbol{B} \rangle) > 0$, then becomes $\eta \langle \boldsymbol{\nabla} \times \boldsymbol{B}' \cdot \boldsymbol{B}' \rangle (\boldsymbol{\nabla} \times \langle \boldsymbol{B} \rangle \cdot \langle \boldsymbol{B} \rangle) < 0$, i.e., as first noted in Ref. [20], the current helicities in the fluctuating and the mean magnetic fields must have opposite signs.

Assume now that the α effect really overcomes the dissipative term in Eq. (19), i.e., $\mathcal{E} \cdot (\nabla \times \langle \boldsymbol{B} \rangle) > \eta (\nabla \times \langle \boldsymbol{B} \rangle)^2$. By using Eq. (18) and the Schwarz inequality $(\nabla \times \langle \boldsymbol{B} \rangle)^2 \langle \boldsymbol{B} \rangle^2 \geq (\nabla \times \langle \boldsymbol{B} \rangle \cdot \langle \boldsymbol{B} \rangle)^2$ one then finds as a necessary condition for the growth of $\langle \boldsymbol{B} \rangle^2$ (cf. Ref. [18])

$$- \langle \nabla \times \boldsymbol{B}' \cdot \boldsymbol{B}' \rangle (\nabla \times \langle \boldsymbol{B} \rangle \cdot \langle \boldsymbol{B} \rangle) > (\nabla \times \langle \boldsymbol{B} \rangle \cdot \langle \boldsymbol{B} \rangle)^2. \tag{20}$$

That is, the current helicity of the fluctuations must exceed that of the mean field by modulus.

Condition (20) has an implication for the evolution of the mean-field magnetic helicity: Since $|\eta \langle \nabla \times \boldsymbol{B}' \cdot \boldsymbol{B}' \rangle| = |\mathcal{E} \cdot \langle \boldsymbol{B} \rangle|$ due to the assumed stationarity and homogeneity of the fluctuations, $|\eta \nabla \times \langle \boldsymbol{B} \rangle \cdot \langle \boldsymbol{B} \rangle| < |\mathcal{E} \cdot \langle \boldsymbol{B} \rangle|$. Then according to Eq. (16) helicity is accumulated in the mean magnetic field, with sign given by the sign of $\mathcal{E} \cdot \langle \boldsymbol{B} \rangle$, i.e. by the sign of α.

Finally, a remark concerning the assumed homogeneity of the fluctuations — or the neglect of the helicity transport, respectively — seems in order. In the RFP, for instance, where poloidal magnetic fields are externally supplied (thus the RFP dynamo is not really a dynamo), helicity transport may dominate over helicity dissipation. So relations have been derived here [6] which, in contrast to Eq. (18), express α in terms of a helicity flux.

Also for astrophysical dynamos helicity transport may be non-negligible. This does not affect the result that the α effect, which is a local effect, corresponds to the simultaneous generation of magnetic helicities in the mean field and in the fluctuations. But the dynamo region may be small and a significant part of the magnetic helicity generated there may be transported out of this region, rather than being dissipated *in situ* (so that the dissipation region is much larger than the generation region). In the case of the sun, for example, the dynamo works in the convection zone, or even in a thin layer at the bottom of the convection zone [21–24], while helicity probably generated by the dynamo is observed in the solar atmosphere [25–27] and also in interplanetary magnetic clouds ejected from the sun [28]. It will be interesting to carry out improved helicity measurements in the solar atmosphere as well as in the solar wind and to analyze them with respect to signatures of the two helicities (that in the mean field and that in the fluctuations) generated by the α effect.

III. CONCLUSION

The α effect corresponds to the simultaneous generation of magnetic helicities in the mean field and in the fluctuations, the generation rates being equal in magnitude and opposite in sign. The mean total magnetic helicity is not influenced by the α effect. This may equally be considered as a transfer of magnetic helicity between the fluctuating and the mean fields, mediated by the α effect. In the case of statistically stationary and homogeneous fluctuations, in particular, the α effect can increase the energy in the mean magnetic field only under the condition that also magnetic helicity is accumulated there. Generally, the two helicities generated by the α effect, that in the mean field and that in the fluctuations, have either to be dissipated in the generation region or to be transported out of this region. The latter, as, for example, the ejection of magnetic helicity from the dynamo region of the sun into the solar atmosphere and the interplanetary space, may provide valuable information on dynamo processes inaccessible to *in situ* measurements.

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